Warm-Up

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CST/CAHSEE: Algebra 2 24.0

Given $f(x) = -x^2 + 3x - 5$, find the function value when $x = 4$.

Review: Algebra 2 8.0

Solve for x using two methods:

$3x^2 + 7x - 20 = 0$

Which method do you find easier?

x

Current: Calculus 4.0

Power Rule for Derivatives

Given: $f(x) = x^n$, the derivative of $f(x)$ is

$$f'(x) = nx^{n-1}$$

Use the power rule to find the derivative of the following function:

$f(x) = 2x^4 + 7x^3 - 21x^2 + 4x - 7$

Other: Calculus 4.0

Recall that a critical number of a function is a value c such that $f'(c) = 0$ or $f'(c)$ DNE.

Find the critical numbers of the following function:

$f(x) = 2x^3 - 6x^2 - 18x + 51$

The Extreme Value Theorem

Objective: Find the absolute extrema of a function on a closed interval.

Standards: Algebra 5.0, 11.0, 14.0, Calculus 3.0

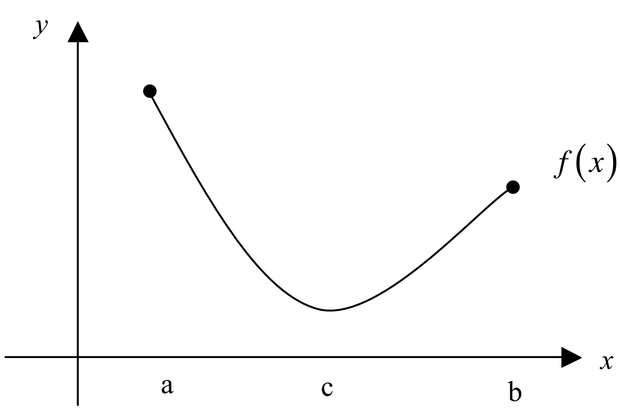
Many functions do not have an absolute minimum value or absolute maximum value over their entire domain but will have absolute extrema on a closed interval. The Extreme Value Theorem states the existence of absolute extrema on closed intervals.

Vocabulary Note: Extrema is plural, while extremum is singular.

> **The Extreme Value Theorem:** If f is continuous on a closed interval $[a,b]$ then f has both a minimum and a maximum on the closed interval $[a,b]$.

The candidates for absolute extrema are the endpoints of the closed interval $[a,b]$ and the critical numbers of the function that lie in the given closed interval.

Let's take a look at the following graph of a function on a closed interval $[a,b]$:



For this function we can see that the absolute maximum occurs at the endpoint, $x = a$, on the closed interval.

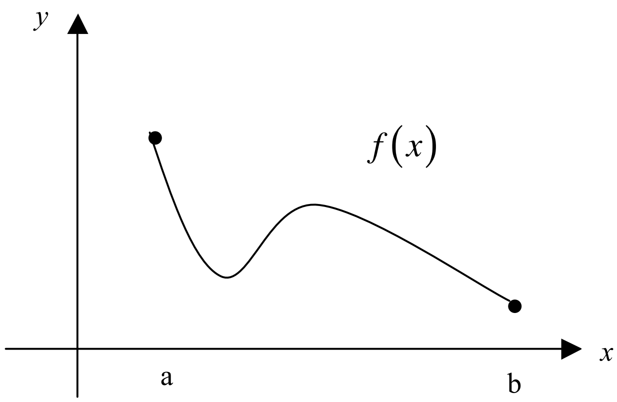
The absolute maximum value is $f(a)$.

Notice that the absolute minimum occurs at the critical number $x = c$ in the interior of the closed interval $[a,b]$.

The absolute minimum value is $f(c)$.

Note: An absolute extreme can also be a relative extreme. On a closed interval, we will refer to relative extreme values as absolute extreme values if they are in fact an absolute minimum or absolute maximum on the interval.

Your Turn: Use the graph to determine the absolute extrema of the function and where the extrema occur.



Does this function have critical numbers? If so, how many on the closed interval $[a,b]$ **?**

Answers: The absolute maximum value is $f(a)$. The absolute maximum occurs at $x=a$.

The absolute minimum value is $f(b)$. The absolute minimum occurs at $x=b$.

There are two critical numbers of f on the closed interval $[a,b]$.

Example: Given: $f(x) = 4x^3 + 15x^2 - 18x + 7,\ x \in [-1,4]$

a) Determine whether the Extreme Value Theorem applies.
If the theorem applies, identify the candidates for absolute extrema then continue to parts b – d.

b) Find the absolute maximum value of f on the given closed interval.

c) At what x-value(s) does the absolute minimum occur on the given closed interval?

d) Find the point(s) where f has an absolute minimum.

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a) f is a polynomial $\therefore f$ is continuous on the closed interval $[-1,4]$ $\therefore$ The Extreme Value Theorem applies. The candidates for absolute extrema are the endpoints of the given interval and the critical numbers of f that lie in the given interval.

$f(x) = 4x^3 + 15x^2 - 18x + 7,\ x \in [-1,4]$

$f'(x) = 12x^2 + 30x - 18$

$f' = 0$ when $12x^2 + 30x - 18 = 0$ Note: f' exists over the Real numbers

$12x^2 + 30x - 18 = 0$

$6(2x^2 + 5x - 3) = 0$

$6(2x-1)(x+3) = 0$

$(2x-1)(x+3) = 0$

$2x - 1 = 0 \qquad x + 3 = 0$

$2x = 1 \quad \text{or} \quad x = -3$

$x = \frac{1}{2}$

$x = -3$ does not lie in the closed interval $[-1,4]$ $\therefore$ $x = -3$ is not a candidate for absolute extrema.

The candidates for absolute extrema are: $x = -1,\ x = \frac{1}{2},$ and $x = 4$.

For parts b and c of the question, test the candidates for absolute extrema to determine which candidate gives the largest function value and which candidate gives the smallest function value. Let's set up a table.

Candidate for Absolute Extrema	$f(x)=4x^3+15x^2-18x+7$	Function Value
$x=-1$	$f(-1)=4(-1)^3+15(-1)^2-18(-1)+7$ $=4(-1)+15(1)+18+7$ $=-4+15+18+7$ $=11+18+7$ $=29+7$ $=36$	$f(-1)=36$
$x=\frac{1}{2}$	$f\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^3+15\left(\frac{1}{2}\right)^2-18\left(\frac{1}{2}\right)+7$ $=4\left(\frac{1}{8}\right)+15\left(\frac{1}{4}\right)-9+7$ $=\frac{1}{2}+\frac{15}{4}-9+7$ $=\frac{2}{4}+\frac{15}{4}-\frac{36}{4}+\frac{28}{4}$ $=\frac{2+15-36+28}{4}$ $=\frac{17-36+28}{4}$ $=\frac{-19+28}{4}$ $=\frac{9}{4}$	$f\left(\frac{1}{2}\right)=\frac{9}{4}$ $f\left(\frac{1}{2}\right)=2\frac{1}{4}$
$x=4$	$f(4)=4(4)^3+15(4)^2-18(4)+7$ $=4(64)+15(16)-72+7$ $=256+240-72+7$ $=496-72+7$ $=424+7$ $=431$	$f(4)=431$

b) By the Extreme Value Theorem, the absolute maximum value of f on the closed interval $[-1,4]$ is 431.

c) By the Extreme Value Theorem, on the closed interval $[-1,4]$ the absolute minimum of f occurs at $x=\frac{1}{2}$.

d) The absolute minimum of f is at the point $\left(\frac{1}{2},\frac{9}{4}\right)$.

<u>Note:</u> Students in Calculus are expected to answer a free-response question using complete sentences and are often asked to justify their answers or conclusions. In cases involving the use of the Extreme Value Theorem, setting up a table or chart to show that all candidates for extrema have been considered will suffice as a justification.

The answers above to parts b – d are examples of a clear, complete sentence as an answer to the proposed questions.

On the following page is a classroom activity that can be used to check for understanding of the theorem. Students should be informed of the use of the table or chart to test candidates and to write complete sentences for their answers or conclusions.

Solutions to the Warm-Up and Classroom Activity can be found at the end of the lesson.

Classroom Activity: Set up partners in the classroom. Partner B will be the person whose birthday is closest to the current date. The other person is Partner A. Each pair will be given a function. Each function will then have a question for Partner A and a question for Partner B. When time is called have one partner share their work with the other then they will switch and the other partner will share. Cut in strips and give one strip at a time.

1) Given: $f(x) = x^3 - 12x,\ x \in [0,4]$ Remember to determine if the Extreme Value Theorem applies.

Partner A: Find the x-value(s) at which the absolute minimum occurs on the given closed interval.

Partner B: Find the absolute maximum value of f on the given closed interval.

2) Given: $f(x) = x^3 - 3x^2,\ x \in [1,3]$ Remember to determine if the Extreme Value Theorem applies.

Partner A: What is the absolute maximum of f on the given closed interval?

Partner B: At what x-value(s) does the absolute minimum of f occur on the given closed interval?

3) Given: $f(x) = x^4 - 4x^2,\ x \in \left[-1,\tfrac{1}{2}\right]$ Remember to determine if the Extreme Value Theorem applies.

Partner A: Find the point(s) where f has an absolute minimum on the given closed interval.

Partner B: Find the point(s) where f has an absolute maximum on the given closed interval.

Solutions to Warm-Up

QI) Solve using two methods: $3x^2 + 7x - 20 = 0$

Method 1: Solve by factoring:

$$3x^2 + 7x - 20 = 0$$

$$(3x-5)(x+4) = 0$$

$$3x - 5 = 0 \qquad\qquad x + 4 = 0$$

$$3x = 5 \qquad\qquad x + 4 - 4 = 0 - 4$$

$$\frac{3x}{3} = \frac{5}{3} \quad \text{OR} \quad x = -4$$

$$x = \frac{5}{3}$$

Method 2: Quadratic Formula: $a = 3,\ b = 7,\ c = -20$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-20)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 + 240}}{6}$$

$$x = \frac{-7 \pm \sqrt{289}}{6}$$

$$x = \frac{-7+17}{6} \quad \text{or} \quad x = \frac{-7-17}{6}$$

$$x = \frac{10}{6} \quad \text{or} \quad x = \frac{-24}{6}$$

$$x = \frac{2 \bullet 5}{2 \bullet 3} \quad \text{or} \quad x = \frac{-1 \bullet 2 \bullet 2 \bullet 2 \bullet 3}{2 \bullet 3}$$

$$x = \frac{5}{3} \quad \text{or} \quad x = -4$$

Method 1 should be easier!

QII) Given $f(x) = -x^2 + 3x - 5$, find the function value when $x = 4$.

$$\begin{aligned} f(4) &= -(4)^2 + 3(4) - 5 \\ &= -16 + 12 - 5 \\ &= -4 - 5 \\ &= -9 \end{aligned}$$

QIII) Use the Power Rule for Derivatives to find the derivative of $f(x) = 2x^4 + 7x^3 - 21x^2 + 4x - 7$.

$$f(x) = 2x^4 + 7x^3 - 21x^2 + 4x - 7$$

$$f(x) = 2x^4 + 7x^3 - 21x^2 + 4x^1 - 7x^0$$

$$f'(x) = 4 \cdot 2x^{4-1} + 3 \cdot 7x^{3-1} - 2 \cdot 21x^{2-1} + 1 \cdot 4x^{1-1} - 0 \cdot 7x^{0-1}$$

$$f'(x) = 8x^3 + 21x^2 - 42x + 4$$

QIV) Find the critical numbers of the function $f(x) = 2x^3 - 6x^2 - 18x + 51$.

$$f'(x) = 6x^2 - 12x - 18$$

$f' = 0$ when $6x^2 - 12x - 18 = 0$ Note: f' exists over the Real numbers

$$6x^2 - 12x - 18 = 0$$

$$6(x^2 - 2x - 3) = 0$$

$$6(x-3)(x+1) = 0$$

$$(x-3)(x+1) = 0$$

$$x - 3 = 0 \text{ or } x + 1 = 0$$

$$x - 3 + 3 = 0 + 3 \text{ or } x + 1 - 1 = 0 - 1$$

$$x = 3 \text{ or } x = -1$$

Since f is a polynomial the domain is the set of all Real Numbers $\therefore$ both $x = 3$ and $x = -1$ are critical numbers of the function f.

Solutions to Classroom Activity

Note: Both partners should verify that the Extreme Value Theorem applies then identify and test the candidates for absolute extrema for all three problems in the activity.

1) Given: $f(x) = x^3 - 12x,\ x \in [0,4]$

f is a polynomial $\therefore f$ is continuous on the closed interval $[0,4]$ $\therefore$ The Extreme Value Theorem applies. The candidates for absolute extrema are the endpoints of the given interval and the critical numbers of f that lie in $[0,4]$.

$f'(x) = 3x^2 - 12$

$f' = 0$ when $3x^2 - 12 = 0$ Note: f' exists over the Real numbers

$$3x^2 - 12 = 0$$

$$3(x^2 - 4) = 0$$

$$3(x-2)(x+2) = 0$$

$$(x-2)(x+2) = 0$$

$x - 2 = 0$ or $x + 2 = 0$

$x = 2$ $\quad$ $x = -2$

$x = -2$ does not lie in the closed interval $[0,4]$ $\therefore$ $x = -2$ is not a candidate for absolute extrema.

The candidates for absolute extrema are: $x = 0,\ x = 2,$ and $x = 4$.

Candidate for Absolute Extrema	$f(x)=x^3-12x$	Function Value
$x=0$	$f(0)=(0)^3-12(0)$ $=0$	$f(0)=0$
$x=2$	$f(2)=(2)^3-12(2)$ $=8-24$ $=-16$	$f(2)=-16$
$x=4$	$f(4)=(4)^3-12(4)$ $=64-48$ $=16$	$f(4)=16$

Partner A: The absolute minimum of $f(x)=x^3-12x$ on the closed interval $[0,4]$ occurs at $x=2$.

Partner B: The absolute maximum value of $f(x)=x^3-12x$ on the closed interval $[0,4]$ is 16.

2) Given: $f(x)=x^3-3x^2,\ x\in[1,3]$

f is a polynomial $\therefore f$ is continuous on the closed interval $[1,3]$ $\therefore$ The Extreme Value Theorem applies. The candidates for absolute extrema are the endpoints of the given interval and the critical numbers of f that lie in $[1,3]$.

$f'(x)=3x^2-6x$

$f'=0$ when $3x^2-6x=0$ Note: f' exists over the Real numbers

$3x^2-6x=0$

$3x(x-2)=0$

$3x=0$ or $x-2=0$

$x=0$ $x=2$

$x=0$ does not lie in the closed interval $[1,3]$ $\therefore$ $x=0$ is <u>not</u> a candidate for absolute extrema.

The candidates for absolute extrema are: $x=1$, $x=2$, and $x=3$.

Candidate for Absolute Extrema	$f(x)=x^3-3x^2$	Function Value
$x=1$	$f(1)=(1)^3-3(1)^2$ $=1-3(1)$ $=1-3$ $=-2$	$f(1)=-2$
$x=2$	$f(2)=(2)^3-3(2)^2$ $=8-3(4)$ $=8-12$ $=-4$	$f(2)=-4$
$x=3$	$f(3)=(3)^3-3(3)^2$ $=27-3(9)$ $=27-27$ $=0$	$f(3)=0$

Partner A: The absolute maximum value of $f(x)=x^3-3x^2$ on the closed interval $[1,3]$ is 0.

Partner B: The absolute minimum of $f(x)=x^3-3x^2$ on the closed interval $[1,3]$ occurs at $x=2$.

3) Given: $f(x) = x^4 - 4x^2, \; x \in \left[-1, \tfrac{1}{2}\right]$

f is a polynomial $\therefore f$ is continuous on the closed interval $\left[-1, \tfrac{1}{2}\right]$ $\therefore$ The Extreme Value Theorem applies. The candidates for absolute extrema are the endpoints of the given interval and the critical numbers of f that lie in $\left[-1, \tfrac{1}{2}\right]$.

$f'(x) = 4x^3 - 8x$

$f' = 0$ when $4x^3 - 8x = 0$ Note: f' exists over the Real numbers

$$4x^3 - 8x = 0$$

$$4x(x^2 - 2) = 0$$

$$4x = 0 \quad \text{or} \quad x^2 - 2 = 0$$

$$x = 0 \qquad\qquad x^2 = 2$$

$$\sqrt{x^2} = \sqrt{2}$$

$$|x| = \sqrt{2}$$

$$x = -\sqrt{2} \;\text{ or }\; x = \sqrt{2}$$

$x = -\sqrt{2}$ does not lie in the closed interval $\left[-1, \tfrac{1}{2}\right]$ $\therefore$ $x = -\sqrt{2}$ is not a candidate for absolute extrema. $x = \sqrt{2}$ does not lie in the closed interval $\left[-1, \tfrac{1}{2}\right]$ $\therefore$ $x = \sqrt{2}$ is not a candidate for absolute extrema. The candidates for absolute extrema are: $x = -1$, $x = 0$, and $x = \dfrac{1}{2}$.

Candidate for Absolute Extrema	$f(x)=x^4-4x^2$	Function Value
$x=-1$	$f(-1)=(-1)^4-4(-1)^2$ $=1-4(1)$ $=1-4$ $=-3$	$f(-1)=-3$
$x=0$	$f(0)=(0)^4-4(0)^2$ $=0-4(0)$ $=0$	$f(0)=0$
$x=\frac{1}{2}$	$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^4-4\left(\frac{1}{2}\right)^2$ $=\frac{1}{16}-4\left(\frac{1}{4}\right)$ $=\frac{1}{16}-1$ $=\frac{1}{16}-\frac{16}{16}$ $=\frac{1-16}{16}$ $=-\frac{15}{16}$	$f\left(\frac{1}{2}\right)=-\frac{15}{16}$

Partner A: The point where $f(x)=x^4-4x^2$ has an absolute minimum on the closed interval $\left[-1,\frac{1}{2}\right]$ is $(-1,-3)$.

Partner B: The point where $f(x)=x^4-4x^2$ has an absolute maximum on the closed interval $\left[-1,\frac{1}{2}\right]$ is $(0,0)$.